On the *Unfolding* of Prisms

Honors Project in Mathematics at Moravian College

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BRIEF ABSTRACT

Geometry is one of the most widely studied subjects in human history. Plane figures, called **polygons**, have been sketched and investigated for millennia. And polygons have three-dimensional analogs called **polyhedra**, which have been the subject of much inquiry since at least the time of the Greek philosophers. More recently (since about the 16th Century C.E.), the idea that polyhedra can be "unfolded" has been under investigation by mathematicians. These unfoldings, called **nets**, have become a standard way to express a three-dimensional polyhedron in two-dimensional Euclidean space. However, whether or not certain classes of polyhedra can be expressed as nets which do not overlap themselves is currently unknown, and this is where my research comes in. It has been proven that some classes of polyhedra always have such a net. Conversely, for some other types of polyhedra, no such nets exist. Some polyhedra which are not **edge-unfoldable**, however, are **vertex-unfoldable** or **generally unfoldable**, two less-strict definitions of unfoldability. The goal of my research is to identify whether a few more classes of polyhedra can be unfolded through one of these methods. I've shown that all genus-0 prisms, convex or not, can be unfolded given certain conditions. I've also shown that certain prisms with genus can be unfolded given certain conditions, or can be made to be generally unfoldable via a few distinct methods.

GENUS-0 RIGHT PRISMS

Genus-0 right prisms are prisms with no holes, whose side faces meet its bases at 90-degree angles. These prisms can be split up into two categories—convex and nonconvex.

Convex, genus-0 right prisms are the most easily unfolded prism, and they are always edge-unfoldable. As we can see in Figure 2, these prisms can simply "splay out" their side faces like the petals of a flower, and then simply attach the top base onto one of these unfolded side faces.

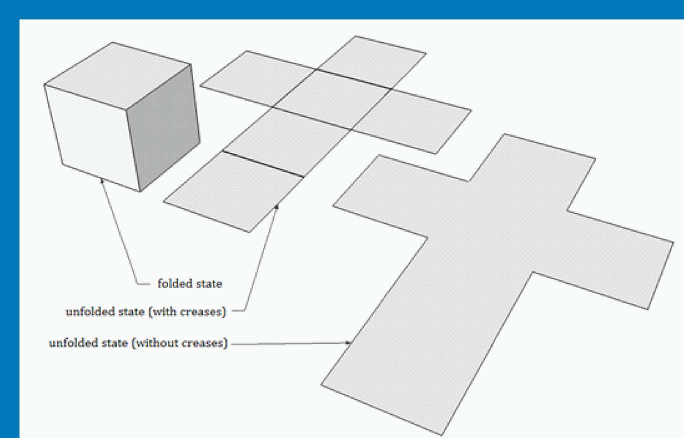

Figure 2

Nonconvex, genus-0 right prisms are only slightly more difficult to unfold. Since a side face and a base of a right prism meet at a 90-degree angle, in order to fold out flat, we need at least one unobstructed external angle of 90-degree measure or greater. The first side face (called the **initial unfolded side face** in my work) folds out flat, and all of the others unfurl in a straight line connected to the initial face, since they're attached at their edges (see Figure 3). The top base can then flip over through the third dimensions in order to lie flat in the plane, attached to the final side face. If a nonconvex, genus-0 right prism does not have at least one unobstructed external angle of ≥ 90 degrees, it is not edge-unfoldable.

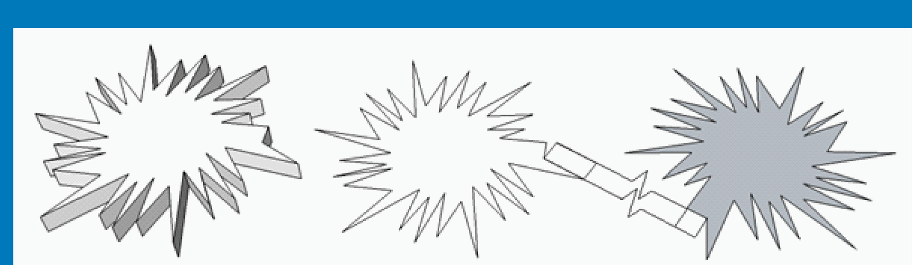

Figure 3

DEFINITIONS

When most people hear the word "prism", they think of the pieces of glass used to split light into its constituent colors, but a mathematical definition of "prism" is slightly more rigorous. A **prism** in mathematics is a polyhedron defined by two congruent, parallel faces called **bases**, which are connected by parallelograms. In my thesis, for simplicity, I called these parallelograms **side faces**.

A **right prism** is a prism whose side faces connect to its bases at 90-degree angles ("right" angle = "right" prism). A prism whose side faces do not meet its end faces at 90-degree angles is called an **oblique prism**. And a **genus-0 prism** (also called a **prism without genus** in my work) is a prism which does not have any "holes" running through it. Since a prism must have congruent side faces, the only way a prism could have holes is if its bases were to have holes in them. A prism with holes is called a **prism with genus**, or a genus-X prism, where X is the number of holes in the prism.

A **convex** polygon is one which has no exterior angles of less than 180 degrees. Similarly, a convex polyhedron is one which has no faces which meet at an angle of less than 180 degrees. A **nonconvex** polygon or polyhedron is one which does not satisfy these properties. The unfoldability of convex polyhedra is understood much more fully than the unfoldability of nonconvex polyhedra.

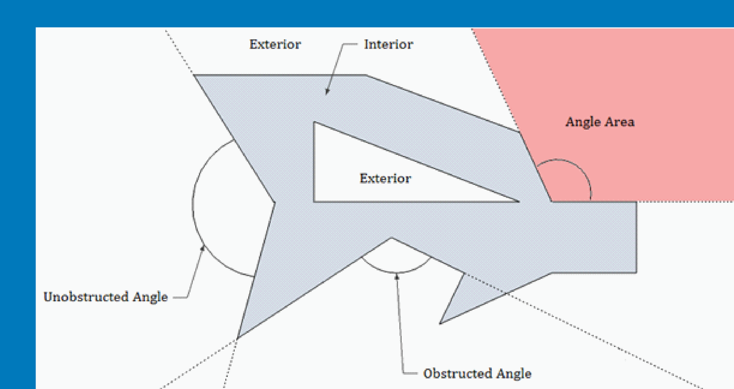

Figure 1

An **unobstructed angle** is also an important concept in my research. This is an angle whose **angle area** does not encompass any part of any polygon. An angle which does not satisfy this condition is an **obstructed angle**.

OBLIQUE PRISMS

An oblique prism is just like a right prism except for one thing—its side faces do not meet the bases at 90-degree angles. The angle at which they do meet is defined by the formula:

$$\theta = \cos^{-1}(\sin\beta \sin\alpha)$$

In the above formula, θ is the angular difference between the normal to the plane of the side face in question and the normal of the bottom base, β is the angle of skew (the angle created by a line connecting two congruent points on the bases, and the projection of that line in the plane of the bottom base), and α is the angular difference between the bottom edge of an arbitrary initial side face, and the bottom edge of the side face in question.

Like right prisms with genus, oblique prisms are all nonconvex, as can be inferred from the above formula. They unfold exactly like right prisms except they need unobstructed external angles of [180 degrees – θ] in order to fold out (where θ depends on the chosen initial side face). The same goes for oblique prisms with genus—simply replace every requirement of ≥ 90 degrees with ≥ [180 degrees – θ].

GENUS-1+ RIGHT PRISMS

By definition, all genus-1+ polyhedra are nonconvex, since at least two of the side faces which make up the interior of the hole necessarily meet at angles that are less than 180 degrees, in order to form a closed hole.

In order to be edge-unfoldable, a prism with genus must also follow the same requirements of edge-unfoldability as a prism without genus—it must have at least one external, unobstructed angle of ≥ 90 degrees. Note that, by definition, the angles which make up the interior of any holes are exterior angles, they are always obstructed, and so the initial unfolded side face can never be an inside side face of a hole.

A genus-1+ right prism is *only* edge-unfoldable if all of its holes have even numbers of edges which all meet at angles of 90 degrees or greater. The holes must have even numbers of edges so that the inside side faces can alternate sides in the unfolded state, and the inside side faces of the holes must all meet at angles of ≥ 90 degrees since they *all* must fold flat *from the bases*. And of course, this only works for a prism of limiting height, which is case-specific.

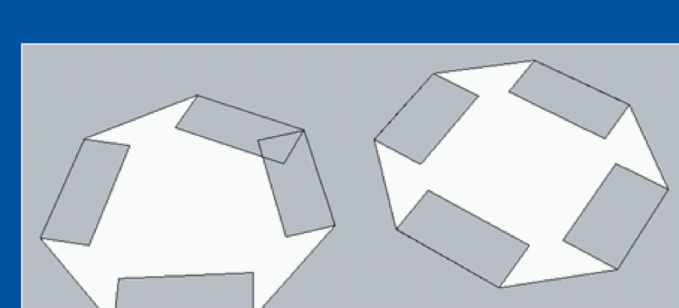

Figure 4

There is also strong evidence that genus-1+ right prisms are also generally unfoldable, meaning that, if we are allowed to make any "cuts" along the bases of the prism (sectioning the single polygon into several), we can unfold any of these prisms. I currently have no *proof* that this works, but I have a method and a rough formula, which shows that for a case as simple as a triangular prism with three triangular holes, there are *over 100,000 distinct possible unfoldings*. Whether or not these potential unfoldings work without overlapping is not yet calculable, however. This formula is as follows:

$$H_{low} = h \cdot \prod_{i=1}^{\alpha-1}\left[2m_i\left(n + \sum_{0}^{i-1} m_i\right)\right]$$

In the above formula, H_low is a *lower bound* on the number of potential unfoldings of a given prism (this formula does not yet take into account a few factors, which could increase the number of unfoldings dramatically), h is the number of possible initial unfolded side faces, α is the number of holes in the prism, m α is the number of side faces of the α-th hole, and n is the number of sides of the prism proper (i.e. not including inside side faces of holes). The geometrically simplest possible polygonal (non-curved) prism—a genus-0 right triangular prism—has no fewer than 90 distinct edge unfoldings, suggesting that the formula above is underestimating by at least of a factor of about 2.

FURTHER RESEARCH

There are many avenues for future research in this field. As can be seen in the chart below, my research primarily focused on nonconvex prismatoids, specifically prisms and oblique prisms. Nonconvex polyhedra are the least understood in terms of unfoldability, and any small classes of these polyhedra that can be proven to have nets—or proven not to—are steps toward understanding unfoldability in general.

Types of Unfoldings for Various Polyhedra					Edge	Vertex	General
Triangulated						YES	YES
Non-Triangulated	Non-Convex				NO		
Non-Triangulated	Non-Convex	Prismatoids	Prismoids, etc.	Smooth			
Non-Triangulated	Non-Convex	Prismatoids	Prisms, Oblique Prisms, Domes, Pyramids	Smooth	*NO*		*YES?*
Non-Triangulated	Non-Convex	Prismatoids	Prisms, Oblique Prisms, Domes, Pyramids	Polygonal	*NO*		*YES?*
Non-Triangulated	Non-Convex	Prismatoids	Prismoids, etc.	Polygonal			
Non-Triangulated		Orthogonal Polyhedra	Others				YES
Non-Triangulated		Orthogonal Polyhedra	Orthogonal Terrains		YES	YES	YES
Non-Triangulated		Orthogonal Polyhedra	Rectangular Prisms		YES	YES	YES
Non-Triangulated	Convex	Prismatoids	Prismoids, etc.	Polygonal			YES
Non-Triangulated	Convex	Prismatoids	Prisms, Oblique Prisms, Domes, Pyramids	Polygonal	YES	YES	YES
Non-Triangulated	Convex	Prismatoids	Prisms, Oblique Prisms, Domes, Pyramids	Smooth	YES	YES	YES
Non-Triangulated	Convex	Prismatoids	Prismoids, etc.	Smooth	YES	YES	YES
Non-Triangulated	Convex						YES

Figure 5